Decomposition of the diagonal and new stable birational invariants

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Definition: (*Rationality and unirationality*)

• A variety X (say, over $\mathbb{C}$) is unirational if there is a dominating rational map $\phi : \mathbb{P}^n \dashrightarrow X$. One can always assume $n = \dim X$.
• X is rational if there is a birational map (i.e. a dominant rational map with an inverse which is a rational map) $\mathbb{P}^n \dashrightarrow X$.

• If X is unirational, it is "**rationally connected**": *through any x, $y \in X$ there exists a rational curve in X, passing through x and y.*

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- If X is unirational, it is "**rationally connected**": *through any* x, $y \in X$ *there exists a rational curve in* X, *passing through* x *and* y.

Definition: (*Stably rational varieties*)

- X is stably rational if $X \times \mathbb{P}^r$ is rational for some r.

Obvious implications:

Rational $\Rightarrow$ Stably rational $\Rightarrow$ Unirational $\Rightarrow$ Rationally connected.

Theorem (Castelnuovo)

If $\dim X = 1$ or $\dim X = 2$ and X is rationally connected, then X is rational (in particular X is unirational).

Rational $\Rightarrow$ Stably rational $\Rightarrow$ Unirational $\Rightarrow$ Rationally connected

• The general question is how strict are these implications. The most important one (and completely open) is:

Conjecture

There exist rationally connected varieties which are not unirational.

The other implications are all strict:

• There exist stably rational non-rational varieties (Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer).

• There exist unirational non-rational varieties (Clemens-Griffiths, Iskovskikh-Manin).

• There exist unirational varieties which are not stably rational (Artin-Mumford, Colliot-Thélène-Ojanguren).

- **Iskovskikh-Manin.** Consider the group $\mathrm{Bir}(X)$ of birational self-maps $X \dashrightarrow X$. If X is rational, this group is enormous (the Cremona group).

Theorem (Iskovskikh-Manin)

Certain smooth quartic hypersurfaces $X \subset \mathbb{P}^4_{\mathbb{C}}$ are unirational. Any such X has $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ (a finite group), hence is not rational.

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• **Artin-Mumford.** Let X be a desingularization of $Y=$ double cover of $\mathbb{P}^3$ with equation $u^2 = f(x_0, \dots, x_3)$, where $\deg f = 4$ and the quartic $f = 0$ has 10 nodes in special position. X is unirational.

Theorem (Artin-Mumford)

(i) X as above has nonzero 2-torsion in $H^3(X, \mathbb{Z})$.
(ii) A stably rational smooth projective variety has no torsion in $H^3(X, \mathbb{Z})$. Hence X as above is not stably rational.

To get (ii): we can assume X rational; by assumption, there is a degree 1 morphism $Y \to X$, where Y is a blow-up of $\mathbb{P}^n_{\mathbb{C}}$, so $H^3(Y, \mathbb{Z})$ has no torsion and $H^3(X, \mathbb{Z}) \hookrightarrow H^3(Y, \mathbb{Z})$.

• Jacobian of a smooth projective complex curve : as a complex torus $J(C) := H^{1,0}(C)^*/H_1(C,\mathbb{Z})$.
• **Griffiths intermediate Jacobian**: X a smooth projective threefold over $\mathbb{C}$ with no nonzero holomorphic 3-forms. Define $J^3(X) := H^{2,1}(X)^*/H_3(X,\mathbb{Z})$, where $H^3(X,\mathbb{C}) = H^{2,1}(X) \oplus \overline{H^{2,1}(X)}$, $H^{2,1}(X)=$ Betti cohomology classes representable by a closed form of type $(2,1)$.
• The complex tori $J(C)$, $J^3(X)$ are ppav's (hence have a canonical Θ divisor): the polarization on $J^3(X)$ comes from $H_1(J^3(X),\mathbb{Z}) \cong H^3(X,\mathbb{Z})$ and Poincaré unimodular pairing on $H^3(X,\mathbb{Z})$.

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Theorem (Clemens-Griffiths)

(i) If X is rational then $J^3(X)$ is isomorphic as a ppav to a product of Jacobians of curves.

(ii) A smooth cubic hypersurface in $\mathbb{P}^4$ is not rational because it does not satisfy criterion (i).

This is a generalization of the Artin-Mumford invariant introduced by Colliot-Thélène and Ojanguren. X is algebraic over $\mathbb{C}$ so X (or rather its set of points) has two topologies: "us" and "Zar". Continuous map: $\pi : X_{us} \to X_{Zar}$. Let A be an abelian group.

Definition: (*Unramified cohomology*)

- $\mathcal{H}^i(A) := R^i\pi_* A$.(This is sheaf on X_{Zar}).
- $H^i_{nr}(X, A) := H^0(X_{Zar}, \mathcal{H}^i(A))$.

- Vanishes for $i > dim\, X$.
- These are stable birational invariants (Colliot-Thélène-Ojanguren).

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Example

- If $H^{2,0}(X) = 0$, $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}) = \mathrm{Tors}(H^3(X, \mathbb{Z}))$ =Brauer group of X.
- (C-T-V '12) If $\mathrm{CH}_0(X) = \mathbb{Z}$, $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hdg}^4(X, \mathbb{Z})/H^4(X, \mathbb{Z})_{alg}$.

Theorem (Colliot-Thélène-Ojanguren 1988)

There exist 6-dimensional unirational varieties with $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \neq 0$.

- The Clemens-Griffiths method works only in dimension 3.
- The Iskovskikh-Manin criterion does not address stable rationality and does not work on some interesting examples (like quartic double solids).
- The Artin-Mumford invariant detects non stable rationality, but it is deformation invariant. For unirational threefolds, this is the only nonzero unramified cohomology group with torsion coeffts:

Theorem (Voisin 2006)

If X is a uniruled 3-fold, $\mathrm{Hdg}^4(X,\mathbb{Z}) = H^4(X,\mathbb{Z})_{alg}$, so $H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}) = 0$.

Further open problems and search for new criteria

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On the other hand:

Conjecture

(stable) rationality is not a deformation invariant property.

Example

It is generally believed that the very general cubic fourfold is not rational.

- If X is rationally connected over $\mathbb{C}$, $\mathrm{CH}_0(X) = \mathbb{Z}$ (all points are rationally equivalent).
- For $X = \mathbb{P}^n$, much more is true, because even over a non-alg. closed field L, $\mathrm{CH}_0(\mathbb{P}^n_L) = \mathbb{Z}$.
- So if X is rational, or stably rational, for any field L containing $\mathbb{C}$, $\mathrm{CH}_0(X_L) = \mathbb{Z}$.
- When the property above is satisfied, one says that $\mathrm{CH}_0(X)$ is *universally trivial*.

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• Let $L := \mathbb{C}(X)$. The diagonal of $X \times X$, seen over the generic point, provides a L-point $\delta_L \in X_L(L)$.

• If X has universally trivial CH_0, this point is rationally equivalent over L to the constant point $x_L := Spec(L) \times x \in X_L(L)$.

Lemma (Auel-Colliot-Thélène-Parimala 2013)

$\mathrm{CH}_0(X)$ *is universally trivial iff* $\delta_L - x_L = 0$ *in* $\mathrm{CH}_0(X_L)$.

- $Spec(L) \times X \subset X \times X$ is the limit of $U \times X$, $U \subset X$ Zariski open dense subset.
- $x_L \in \mathrm{CH}_0(X_L) = \mathrm{CH}^n(X_L)$ is the restriction of $X \times x$, and $\delta_L \in \mathrm{CH}^n(X_L)$ is the restriction of Δ_X. Hence by localization exact sequence $\delta_L - x_L = 0$ in $\mathrm{CH}_0(X_L)$ iff for some $D \subsetneq X$ closed algebraic,

$$\Delta_X = X \times x + Z \text{ in } \mathrm{CH}^n(X \times X)$$

with Z supported on $D \times X$.
- This is called a Chow-theoretic decomposition of diagonal.

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• This is called a Chow-theoretic decomposition of diagonal.
• A weaker version is the corresponding cohomological decomposition of the diagonal:

$$[\Delta_X] = [X \times x] + [Z] \text{ in } H^{2n}(U \times X, \mathbb{Z}).$$

with Z supported on $D \times X$.
• The existence of such decompositions is a necessary criterion for stable rationality.

Lemma

If X admits a cohomological decomposition of the diagonal, $\mathrm{Tors}(H^3(X,\mathbb{Z}))=0$.
In particular, if X admits a Chow-theoretic decomposition of the diagonal, $\mathrm{Tors}(H^3(X,\mathbb{Z}))=0$.

Lemma

If X admits a cohomological decomposition of the diagonal, $\mathrm{Tors}(H^3(X,\mathbb{Z}))=0$.
In particular, if X admits a Chow-theoretic decomposition of the diagonal, $\mathrm{Tors}(H^3(X,\mathbb{Z}))=0$.

Sketch of proof. Indeed, let $j:\widetilde{D}\to X$ be a desingularization of D, with a lift $\widetilde{Z}\in \mathrm{CH}^n(\widetilde{D}\times X)$. Then

$$(j,Id_X)_*([\widetilde{Z}])=[\Delta_X]-[X\times x] \text{ in } H^{2n}(X\times X,\mathbb{Z}).$$

Hence, for any $\alpha\in H^3(X,\mathbb{Z})$,

$$\alpha=([\Delta_X]-[X\times x])^*\alpha=j_*([\widetilde{Z}]^*\alpha) \text{ in } H^3(X,\mathbb{Z}),$$

with $[\widetilde{Z}]^*\alpha\in H^1(\widetilde{D},\mathbb{Z})$.
But $H^1(\widetilde{D},\mathbb{Z})$ has no torsion, so $\alpha=0$ if $\alpha\in\mathrm{Tors}(H^3(X,\mathbb{Z}))$.

Theorem (Voisin 2013)

Let $\mathcal{X} \to B$ be a flat morphism of relative dim ≥ 2 onto a smooth curve B over $\mathbb{C}$; let $0 \in B$. Assume the general fiber $\mathcal{X}_t$ is smooth and $\mathcal{X}_0$ has at worst ordinary quadratic singularities.
(i) If $\mathcal{X}_t$ has a Chow-theoretic decomposition of diagonal for $t \neq 0$, then so does the desingularization $\widetilde{\mathcal{X}}_0$.
(ii) If $\mathcal{X}_t$ has a cohomological decomposition of diagonal for $t \neq 0$, then so does $\widetilde{\mathcal{X}}_0$, assuming the even degree integral cohomology of $\widetilde{\mathcal{X}}_0$ is algebraic.

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Sketch of proof. For general $t \in B$, there exist $D_t \subset \mathcal{X}_t$, $Z_t \subset D_t \times \mathcal{X}_t$, and $x_t \in \mathcal{X}_t$ such that $\Delta_{\mathcal{X}_t} = \mathcal{X}_t \times x_t + Z_t$ in $\mathrm{CH}(\mathcal{X}_t \times \mathcal{X}_t)$.
We pass to a finite cover B' of B so as to have the data above in family and specialize to $\mathcal{X}_0 \times \mathcal{X}_0$ and this gives a decomp of the diagonal for $\mathcal{X}_0$.
Taking proper transforms in $\widetilde{\mathcal{X}_0} \times \widetilde{\mathcal{X}_0}$, one gets:
$\Delta_{\widetilde{\mathcal{X}_0}} = \widetilde{\mathcal{X}_0} \times x_0 + Z + Z'$ in $\mathrm{CH}(\widetilde{\mathcal{X}_0} \times \widetilde{\mathcal{X}_0})$,
with Z supported on $D_0 \times \widetilde{\mathcal{X}_0}$ and Z' supported on $E \times \widetilde{\mathcal{X}_0} \cup \widetilde{\mathcal{X}_0} \times E$.
Then use that E is a union of quadrics to decompose Z'.

Remark

In dim 3, de Fernex-Fusi have a similar degeneration result for rationality.

Theorem (Voisin 2013)

The very general k-nodal double solid with $k \leq 7$ is not stably rational. (On the other hand, it has no torsion in H^3 by Endrass.)

Remark

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Sketch of proof.

• The Artin-Mumford threefold does not admit a Chow-theoretic decomposition of the diagonal because $\mathrm{Tors}(H^3(X, \mathbb{Z})) \neq 0$ (use the lemma).

• On the other hand, it is the desingularization of a 10-nodal quartic double solid.

• So by the degeneration result, it suffices to show that the very general $k \leq 7$-nodal quartic double solid can be specialized to the Artin-Mumford double solid.

• The degeneration argument also applies to prove that the very general $k \leq 7$-nodal double solid X does not admit a **cohomological** decomposition of the diagonal.

• Recall the ppav $J^3(X) = H^3(X, \mathbb{C})/(H^{2,1}(X) \oplus H^3(X, \mathbb{Z})) = H^{n-1,n-2}(X)^*/H_{2n-3}(X, \mathbb{Z})$ with Abel-Jacobi map

$\Phi_X : \mathrm{CH}^2(X)_{hom} \to J^3(X)$,

$z \mapsto \int_\gamma \in H^{n-1,n-2}(X)^*$, $\partial\gamma = z$.

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Theorem (Bloch-Srinivas)

$\Phi_X : \mathrm{CH}^2(X)_{hom} \to J^3(X)$ *is a group isomorphism for any rationally connected variety.*

• The left hand side is not an algebraic variety (it is a limit of quotients of algebraic varieties by equivalence relation), while the right hand side is an algebraic variety. Φ_X is algebraic in a certain functorial sense.

Question. *Does there exist a universal codimension 2 cycle* $Z \in \mathrm{CH}^2(J^3(X) \times X)$ *ie:*
$\Phi_Z : J^3(X) \to J^3(X)$, $t \mapsto \Phi_X(Z_t)$, *is the identity?*

Theorem (voisin 2012 and 2014)

Let X be a rationally connected threefold. Then X admits a cohomological decomposition of the diagonal iff:
(i) $\mathrm{Tors}(H^3(X, \mathbb{Z})) = 0$.
(ii) There exists a universal codimension 2 cycle on $J^3(X) \times X$.
(iii) There exists a 1-cycle $z \in J^3(X)$ of the minimal class, ie $[z] = \theta^{g-1}/(g-1)!$, $g = dim\, J^3(X)$.

Hence (i), (ii), (iii) are necessary criteria for stable rationality. (i) is Artin-Mumford's criterion. (iii) generalizes Clemens-Griffiths' criterion.

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Corollary

The very general 7-nodal double solid X does not have a universal codimension 2 cycle on $J^3(X) \times X$.

Indeed, $\mathrm{Tors}(H^3(X,\mathbb{Z})) = 0$ by Endrass, and $dim\, J^3(X) = 3$ so $J^3(X)$ is a Jacobian, hence (iii) is satisfied. But X does not admit a cohomological decomposition of the diagonal.

Theorem (Voisin 2014)

Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface of odd dimension or a cubic fourfold. Then X admits a cohomological decomposition of the diagonal iff X admits a Chow-theoretic decomposition of the diagonal.

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Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface of odd dimension or a cubic fourfold. Then X admits a cohomological decomposition of the diagonal iff X admits a Chow-theoretic decomposition of the diagonal.

The proof uses the observation that $X^{[2]}$ is birational to the projective bundle over X with fiber over x the lines in $\mathbb{P}^n$ passing through x, by the map

$(x, y) \mapsto (l_{x,y}, z),\ l_{x,y} =< x, y >,\ z + x + y = l_{x,y} \cap X.$

- Also used by Galkin and Shinder to prove :

Theorem (Galkin and Shinder 2014)

Assume the cancellation conjecture for $K_0(Var)$. Then if a cubic fourfold X is rational, its variety of lines $F(X)$ is birational to $S^{[2]}$, where S is a $K3$ surface.

This would imply in particular that very general cubic fourfolds are not rational.

• Recall $\mathrm{CH}_0(X)$ being universally trivial is equivalent to X admitting a Chow-theoretic decomposition of the diagonal.

Theorem (Voisin 2014)

Let X be a smooth cubic threefold. Then $\mathrm{CH}_0(X)$ is universally trivial iff there exists a 1-cycle on $J^3(X)$ in the class $\theta^4/4!$.

• Whether this is satisfied or not is a very classical open problem.

• There is a countable union of subvarieties of codimension ≤ 3 in the moduli space of X where this is satisfied.

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Theorem (Voisin 2014)

Let X be a cubic fourfold. Assume X is special in the sense of Hassett, with discriminant not divisible by 4. Then $\mathrm{CH}_0(X)$ is universally trivial.

Both results use the previous theorem. One shows that under the assumptions made, X admits a cohomological decomposition of the diagonal, hence a Chow-theoretic one.